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SURGERY ON A CURVE IN A SOLID TORUS

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ABSTRACT. We consider the following surgery question: If a regular neighborhood of a polyhedral knot in a solid torus is removed and then sewn back differently, what manifold results? We consider two classes of knots, torus knots and what we call doubly twisted knots. We obtain some related results on surgery on knots in S^3.

1. **Introduction.** In this paper we consider surgery on a simple closed curve in a solid torus. That is, we consider the following question: If a regular neighborhood of a knot in a solid torus is removed and then sewn back differently, what manifold results? In particular, when is the manifold a solid torus?

A similar kind of surgery, surgery on knots in the 3-sphere, has been extensively studied. For a discussion of the history and importance of this problem, and for other references, see [7], [2], or [8]. Here we consider surgery on two kinds of knots, torus knots and what we call "doubly twisted knots." We also obtain some related results concerning surgery on knots in the 3-sphere.

2. **Definitions and other preliminaries.** In this paper, all manifolds, embeddings, regular neighborhoods, etc., will be piecewise linear. A *solid torus* is a 3-manifold-with-boundary homeomorphic to the product $S^1 \times D^2$ of a circle and a disk. A *torus* is a 2-manifold-without-boundary homeomorphic to the boundary of a solid torus. A *homotopy 3-cell* is a compact simply connected 3-manifold whose boundary is a 2-sphere. A *fake cube* is a homotopy 3-cell which is not a 3-cell. A *homotopy solid torus* is an orientable 3-manifold-with-boundary homeomorphic to a homotopy 3-cell with a pair of disks on its boundary identified. Note that the boundary of a homotopy solid torus is a torus and that if the homotopy 3-cell is a real 3-cell then the homotopy solid torus is a real solid torus. If T is a homotopy solid torus, a *meridian* of T is a simple closed curve on Bd T that bounds a disk in T but is not null-homologous in Bd T. A *longitude* of T is a simple closed curve on Bd T which is transverse to a meridian. A *core* of a solid torus T is a simple closed curve $h(S^1 \times \{a\})$ where h is a homeomorphism of $S^1 \times D^2$ onto T and $a \in \operatorname{Int} D^2$.

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Let T be an unknotted solid torus in S^3. Let (M, L) be a meridian-longitude pair for T such that L bounds a disk in $S^3 - \text{Int } T$. Suppose also K is a polyhedral simple closed curve in Int T, T' is a regular neighborhood of K in Int T and (c, g) is a meridian-longitude pair for T'. If r and s are relatively prime nonzero integers, we shall let $T_g(K: r, s)$ denote the orientable 3-manifold-with-boundary obtained from T by (r, s)-surgery on T'. That is, $T_g(K: r, s)$ is obtained from T by removing Int T' and then sewing a solid torus T'' to $T - \text{Int } T'$ by a homeomorphism of Bd T'' onto Bd T' which takes a meridian of T'' to a curve on Bd T' homologous to $rc + sg$. If g is homologous in $T - \text{Int } T'$ to a multiple of L then we abbreviate $T_g(K: r, s)$ to $T(K: r, s)$. Note that, for a particular knot K in Int T, the family of 3-manifolds

$$\{T_g(K: r, s): (r, s) = 1\}$$

is the same no matter what longitude g is used. For, if g and g' are longitudes transverse to the meridian c, then $g \sim kc + g'$ ($\sim$ means homologous with integer coefficients) for some integer k. Hence, $rc + sg \sim (r + sk)c + sg'$ so that $T_g(K: r, s)$ is topologically equivalent to $T_{g'}(K: r + sk, s)$. We shall use $G(K: r, s)$ to denote the group $\pi_1(T(K: r, s))$. By van Kampen's theorem,

$$G(K: r, s) \cong \pi_1(T - K)/c^r g^s$$

where the right side denotes $\pi_1(T - K)$ modulo the smallest normal subgroup containing $c^r g^s$.

Suppose K is a knot in S^3, (c, g) is a meridian-longitude pair for K and (r, s) is a pair of relatively prime integers. We shall let $M_g^3(K: r, s)$ denote the manifold obtained by removing a regular neighborhood of K from S^3 and sewing in a solid torus whose meridian is sewn to a simple closed curve homologous to $rc + sg$. If $g \sim 0$ in $S^3 - K$ then we write $M^3(K: r, s)$ instead of $M_g^3(K: r, s)$. We say K has *property* P if and only if $\pi_1(M^3(K: r, s))$ is trivial only in the case $(r, s) = (\pm 1, 0)$. It is known that a nontrivial knot K has property P if and only if no counterexample to the Poincaré conjecture can be obtained by surgery on K and the complement of K in S^3 is unique among knot complements. See [2] or [8].

In [2], Bing and Martin considered surgery on certain knots in solid tori and we include their result for completeness.

THEOREM 1. *If K is a knot in the solid torus T such that K is not a core of T and T has as meridional disk which intersects K exactly once, then the surgery manifold $T(K: r, s)$ is not a homotopy solid torus except in the trivial case $(r, s) = (\pm 1, 0)$.*

We conclude this section with a lemma that characterizes homotopy solid tori algebraically. Before stating our lemma, we note the following fact: If m and n are relatively prime, then the group $G = \{a, b: ab = ba, a^m b^n = 1\}$ is infinite cyclic. To show this, let $\{c\}$ be the free group on one generator and define homomorphisms $f: \{c\} \longrightarrow G$ and $g: G \longrightarrow \{c\}$ by $g(a^i b^j) = c^{ni-mj}$ and $f(c) = a^\beta b^{-\alpha}$ where $\alpha m + \beta n = 1$. Then fg and gf are both the identity mappings.

LEMMA 1. *Suppose T is an orientable 3-manifold such that* Bd T *is a torus and $\pi_1(T)$ is infinite cyclic. Then T is a homotopy solid torus.*

PROOF. By the loop theorem, there is a simple closed curve K on Bd T such that K is homotopically nontrivial in Bd T and such that K bounds a disk D whose interior lies in Int T. Let N_1 be a regular neighborhood (in T) of D such that $N_1 \cap \text{Bd } T$ is an annulus. Let $C_1 = \text{Cl}(T - N_1)$. Then Bd C_1 is a 2-sphere. Let N_2 be a regular neighborhood (in C_1) of Bd C_1. Let $C = \text{Cl}(C_1 - N_2)$. Then Bd C is a 2-sphere and we wish to show C is simply connected. Now $T = C \cup (N_1 \cup N_2)$, $C \cap (N_1 \cup N_2) = \text{Bd } C = \text{Bd}(N_1 \cup N_2)$, and $N_1 \cup N_2$ is a regular neighborhood of $D \cup \text{Bd } T$. Also, $\pi_1(\text{Bd } T) = \{a, b: ab = ba\}$; therefore,

$$\pi_1(D \cup \text{Bd } T) = \{a, b: ab = ba, a^m b^n = 1\}$$

where K represents the element $a^m b^n$ of $\pi_1(\text{Bd } T)$. Since K is a simple closed curve, m and n are relatively prime; hence $\pi_1(N_1 \cup N_2) \cong \pi_1(D \cup \text{Bd } T)$ is infinite cyclic. But $(N_1 \cup N_2) \cap C$ is simply connected; therefore $\pi_1(T) \cong \pi_1(N_1 \cup N_2) * \pi_1(C)$, that is, the free product. But $\pi_1(T)$ and $\pi_1(N_1 \cup N_2)$ are both infinite cyclic; hence $\pi_1(C)$ must be trivial. Therefore, C is a homotopy 3-cell and T is a homotopy solid torus.

3. **Surgery on a torus knot in a solid torus.** In this section we begin consideration of surgery on curves in solid tori. Our first case is that of a nicely embedded nontrivial torus knot. Throughout the rest of this paper, T will refer to the standard unknotted solid torus in S^3. In this section, $K(p, q)$ will refer to a (p, q) torus knot nicely embedded in T. That is, we assume there is a polyhedral annulus A in T such that one boundary component of A is $K(p, q)$ and the other boundary component is a simple closed curve K_1 on Bd T which is homologous to $pM + qL$, where M is a meridian of T and L is a longitude of T which is null-homologous in $S^3 - \text{Int } T$. We may assume that $q > 0$. By nontriviality, we mean that K is not a core of T, hence $q \neq 1$. In the next theorem we will use the following construction: Let T' be a regular neighborhood of K in Int T. We will assume that T' is constructed carefully enough

so that $A_1 = A \cap (T - \text{Int } T')$ is an annulus, one of whose boundary components is K_1 and whose other boundary component is $A_1 \cap T' = A_1 \cap \text{Bd } T' = A \cap \text{Bd } T'$. We shall also assume there is a regular neighborhood R of A_1 in $T - \text{Int } T'$ such that $A_1 \cap \text{Bd } R = \text{Bd } A_1$, $R \cap \text{Bd } T = \text{Bd } R \cap \text{Bd } T$ is an annulus with K_1 as a centerline, and $R \cap T' = \text{Bd } R \cap \text{Bd } T'$ is an annulus with $A_1 \cap T'$ as a centerline. In the following theorem we will use the longitude d of T' where d is a boundary component of the annulus $R \cap T'$. Note that $d \sim A_1 \cap T'$ on Bd T' and $d \sim K_1$ in $T - \text{Int } T'$. It should also be noted that d is, in general, not homologous (in $T - \text{Int } T'$) to any multiple of L so that $T_d(K\colon r, s) \neq T(K\colon r, s)$. We shall assume, unless stated otherwise, that the surgery is nontrivial, that is, $(r, s) \neq (\pm 1, 0)$.

THEOREM 2. *If K is a (p, q) torus knot, $q > 1$, embedded in T as described above, then $T_d(K\colon r, s)$ is a union of two solid tori T_1 and T_2 such that $T_1 \cap T_2$ is an annulus B on the boundary of each and B circles* Bd T_1 *q times longitudinally and circles* Bd T_2 *r times longitudinally.*

PROOF. First we recall that $T_d(K\colon r, s) = (T - \text{Int } T') \cup T''$ where $T'' \cap (T - \text{Int } T') = \text{Bd } T'' \cap \text{Bd}(T - \text{Int } T') = \text{Bd } T'$ and a meridian of T'' is sewn to a curve homologous to $rc + sd$. (c is a meridian of T'.) Let $T_1 = \text{Cl}[T - (R \cup T')]$ and $T_2 = R \cup T''$. T_1 is a solid torus since $R \cup T'$ is a regular neighborhood of the curve K_1 on Bd T. Next we show T_2 is a solid torus. Let $B_1 = R \cap T''$. B_1 is an annulus with $A_1 \cap T''$ as a centerline. Now B_1 circles Bd R exactly once longitudinally so R is a regular neighborhood of the simple closed curve $A_1 \cap T''$. Therefore T_2 is T'' plus a regular neighborhood of a curve on Bd T''. Hence T_2 is homeomorphic to T''.

It remains to show that the intersection of T_1 and T_2 is correct. Let $B = T_1 \cap T_2$. Now $B = [R \cup T'] \cap \text{Cl}[T - (R \cup T')] = \text{Cl}[\text{Bd}(R \cup T') - \text{Bd } T]$ which is an annulus since $\text{Bd } T \cap \text{Bd}(R \cup T') = \text{Bd } T \cap \text{Bd } R$ is an annulus. Now d is a centerline of B and I circles T q times longitudinally. Therefore d (and hence B) circles T_1 q times longitudinally, since T_1 is a strong deformation retract of T. Also, there is a homeomorphism of $T_2 = T'' \cup R$ onto T'' which is the identity on $\text{Cl}[\text{Bd } T'' - R]$ and which takes $\text{Cl}[\text{Bd } R - T'']$ onto $R \cap T''$. Since d is a centerline of B and the homeomorphism is the identity on d, it suffices to find the number of times d circles T'' longitudinally. To do this, it suffices to find the (algebraic) intersection number of d and a meridian of T''. Since a meridian of T'' is homologous to $rc + sd$, that intersection number is r. Hence d circles T'' r times longitudinally and so B circles T_2 r times longitudinally. This finishes the proof of Theorem 2.

As previously mentioned, $d \sim K_1 \sim pM + qL$ in $T - \text{Int } T'$. Also,

$M \sim qc$. Therefore $(-pq)c + d \sim -qL$. Hence, if g denotes a longitude of T' which is homologous in $T - \text{Int } T'$ to a multiple of L, $g \sim (-pq)c + d$ or $d \sim pqc + g$. Therefore, $rc + sd \sim rc + spqc + sg$. Hence $T_d(K: r, s) = T(K: r + spq, s)$ and $T(K: r', s') = T_d(K: r' - s'pq, s')$.

COROLLARY 1. *If* $K = K(p, q)$ *then*
(i) $\pi_1(T_d(K: r, s)) = \{w, e: w^r = e^q\}$ *and*
(ii) $G(K: r, s) = \{w, e: w^{r-spq} = e^q\}$.

This corollary follows directly from Theorem 2 by an application of van Kampen's theorem.

COROLLARY 2. $T_d(K: r, s)$ *is a homotopy solid torus if and only if* $r = \pm 1$. $T(K: r, s)$ *is a homotopy solid torus if and only if* $r - spq = \pm 1$.

PROOF. This corollary follows from Lemma 1 and the fact that $\pi_1(T_d(K: r, s))$ is infinite cyclic if $r = \pm 1$ and is nonabelian if $|r| > 1$.

COROLLARY 3. *If* $T_d(K: r, s)$ *or* $T(K: r, s)$ *is a homotopy solid torus then it is a real torus.*

PROOF. If $T_d(K: r, s)$ is a homotopy solid torus then $r = \pm 1$. Thus $T_d(K: r, s) = T_1 \cup T_2$ where $T_1 \cap T_2$ is an annulus which circles Bd T_2 exactly once longitudinally. Hence T_2 is a regular neighborhood of $T_1 \cap T_2$, so $T_1 \cup T_2$ is homeomorphic to T_1.

COROLLARY 4. *If* p *is congruent to* r (mod q) *and* n *is congruent to* q (mod r) *where* m *and* n *are integers such that* $mr + ns = 1$, *then* $T_d(K: r, s)$ *is homeomorphic to a cube with an* (r, q) *torus knotted hole and* $T(K: r, s)$ *is homeomorphic to a cube with an* $(r - spq, q)$ *knotted hole.*

A cube with an (r, q) torus knotted hole is $\text{Cl}[S^3 - N(r, q)]$ where $N(r, q)$ is a regular neighborhood of an (r, q) torus knot. Since the knot lies on the boundary of an unknotted solid torus, the cube with a hole may be written as $T_1^* \cup T_2^*$ where $B^* = T_1^* \cap T_2^* = \text{Bd } T_1^* \cap \text{Bd } T_2^*$ is an annulus which circles Bd T_1^* q times longitudinally and r times meridionally and which circles Bd T_2^* r times longitudinally and q times meridionally.

PROOF OF COROLLARY 4. By Theorem 2, $T_d(K: r, s) = T_1 \cup T_2$ where $T_1 \cap T_2$ is the annulus B with d as a centerline and B circles Bd T_1 q times longitudinally and circles Bd T_2 r times longitudinally. Since T_1 is a solid torus contained in T and a centerline of T_1 is also a centerline of T, there is a meridian-longitude pair (M_1, L_1) for T_1 such that $d \sim pM_1 + qL_1$.

Next we wish to find a meridian-longitude pair for T_2. Let c_2 be a meridian of the solid torus $T' \cup R$ which bounds a disk b_2 in $T' \cup R$ such that $b_2 \cap \text{Bd } T'$ is the meridian c of T'. Also, d is a longitude of $T' \cup R$. Let M'' be a simple closed curve on Bd T'' homologous to $rc + sd$. By construction, M'' is a meridian of T''; hence a simple closed curve M_2 on Bd T_2 which is homologous to $rc_2 + sd$ is a meridian of T_2.

Now there are integers m and n such that $mr + ns = 1$. We claim a simple closed curve L_2 on Bd T_2 such that $L_2 \sim -nc_2 + md$ is a longitude for T_2. First $d \sim nM_2 + rL_2$ and $c_2 \sim mM_2 - sL_2$, so M_2 and L_2 generate $H_1(\text{Bd } T_2)$. Hence it suffices to show that the algebraic intersection number of M_1 and L_1 is 1. Let $\#(J_1, J_2)$ stand for the algebraic intersection number of two oriented curves J_1 and J_2 on a torus. Then, since $M_2 \sim rc_2 + sd$, $\#(d, M_2) = r$ and $\#(c_2, M_2) = -s$ (with appropriate orientations). Hence $\#(L_2, M_2) = \#(-nc + md, M_2) = -n(-s) + mr = 1$.

Now we construct a homeomorphism which takes $T_d(K: r, s)$ onto $T_1^* \cup T_2^*$. By hypothesis, there is an integer n_1 such that $r = p + n_1 q$. Let L'_1 be a simple closed curve on Bd T_1 such that $L'_1 \sim n_1 M_1 + L_1$. Then L'_1 is a longitude of T_1 and $d \sim rM_1 + qL'_1$. Therefore, since d is a centerline of B, there is a homeomorphism f_1 of T_1 onto T_1^* such that $(f_1(M_1), f_1(L'_1))$ is a meridian-longitude pair for T_1^* and $f_1(B) = B^*$. Similarly, there is an integer n_2 such that $n = q + n_2 r$. Let $L'_2 = n_2 M_1 + L_2$. Then $d \sim qM_1 + rL'_2$. Therefore, there is a homeomorphism f_2 of T_2 onto T_2^* such that $(f_2(M_2), f_2(L'_2))$ is a meridian-longitude pair for T_2^* and f_2 agrees with f_1 on B. This concludes the proof of Corollary 4.

LEMMA 2(a). *If* $r - spq = 1$ (*and hence* $T(K(p, q): r, s)$ *is a solid torus*) *then*

$$G(K(p, q): r, s) = \{b, x: bx = xb, x^r b^{sq^2} = 1\},$$

where b *is the element of* $\pi_1(T(K: r, s))$ *generated by the longitude* L *and* x *is the element generated by the meridian* M. *In addition, a simple closed curve homologous to* $rM + sq^2 L$ *is a meridian of* $T(K: r, s)$. *If* $r - spq = -1$ *then the conclusion is the same with* s *replaced by* $-s$.

PROOF. Our first task is to construct a presentation for $\pi_1(T - K(p, q))$ in terms of b, x and y where y is the element of the fundamental group generated by a core of T. Since K is nicely embedded in T, there is a solid torus T^* in Int T such that K lies on Bd T^* and such that a core of T^* is also a core of T.

Let $S = \text{Cl}(T - T^*)$. Then S is homeomorphic to the product of a torus and an interval. Let T' be the regular neighborhood of K as described at the beginning of this section. We shall assume that $T' \cap T^*$, $T' \cap S$ and $T' \cap \operatorname{Bd} T^*$ are regular neighborhoods of K in T^*, S and $\operatorname{Bd} T^*$ respectively. We will find generators and relations for $\pi_1(T - K) \cong \pi_1(\operatorname{Cl}(T - T'))$ from $\pi_1(\operatorname{Cl}(T^* - T'))$ and $\pi_1(\operatorname{Cl}(R - T'))$ using van Kampen's theorem.

Now $\operatorname{Cl}(T^* - T')$ is a solid torus. Let y be the element of $\pi_1(\operatorname{Cl}(T^* - T'))$ generated by a core of T^* which misses T'. Then $\pi_1(\operatorname{Cl}(T^* - T'))$ is a free group with generator y. Also, $\pi_1(\operatorname{Cl}(R - T')) \cong \{x, b\colon bx = xb\}$. Now

$$\operatorname{Cl}(R - T') \cap \operatorname{Cl}(T^* - T') = \operatorname{Cl}(\operatorname{Bd} T^* - T')$$

is an annulus which circles T^* p times meridionally and q times longitudinally. Let z be a generator of $\pi_1(\operatorname{Cl}(\operatorname{Bd} T^* - T'))$. In $\pi_1(\operatorname{Cl}(T^* - T'))$, $z = y^q$ and in $\pi_1(\operatorname{Cl}(R - T'))$, $z = b^q x^p$. Therefore, by van Kampen's theorem,

$$\pi_1(T - K(p, q)) \cong \pi_1(\operatorname{Cl}(T - T')) \cong \{b, x, y\colon bx = xb,\ y^q = b^q x^p\}.$$

Our next task is to find a meridian-longitude pair for $K(p, q)$ in terms of x, y and b. Our method is to use the overcrossing-undercrossing presentation of $T - K(p, q)$ in essentially the same way as J. Hempel did in [6]. Instead of using $\pi_1(T - K(p, q))$ directly, we will use the link group $\pi_1(S^3 - (M^* \cup K(p, q)))$ where M^* is a curve in $S^3 - T$ which is parallel to M.

Now, since p and q are relatively prime, there are integers α and β such that $\alpha p + \beta q = 1$ and $\beta > 0$. Then $\pi_1(S^3 - (M^* \cup K(p, q)))$ has generators $a_1, \cdots, a_{p+q}$ (see Figure 1) and the following relations:

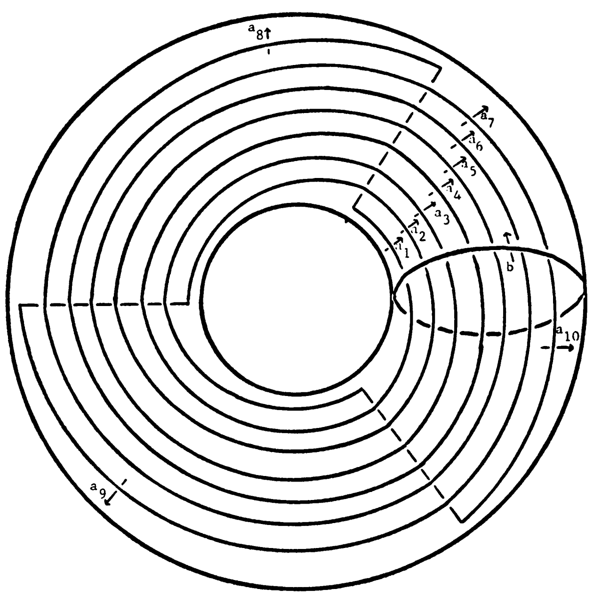

FIGURE 1

(1) for $1 \leqslant k \leqslant p$, $a_k \cdots a_{k+q-1} = a_{k+1} \cdots a_{k+q}$, and

(2) if $k > p$ and $k = np + j$ with $0 \leqslant j < p$ then $a_k = b^n a_j b^{-n}$ where we define $a_0 = a_p$.

Since y is represented by a core of T^*, $y = a_1 \cdots a_p b$ and, since x is represented by a curve parallel to M, $x = a_1 \cdots a_q$. Now the relations in (2) imply

(3) if $1 \leqslant k \leqslant p$ and $np + k \leqslant p + q$ then $a_{np+1} \cdots a_{np+k} = b^n a_1 \cdots a_k b^{-n}$. Define $x_k = a_k \cdots a_{k+q-1}$ for $1 \leqslant k \leqslant p + 1$. Then the relations in (1) are $x_1 = x_2 = \cdots = x_{p+1}$. We wish to write each x_k in terms of $a_1, \cdots, a_p$ and b. From this point, we proceed in two cases.

Case 1. $p < q$. Let $q = jp + \delta$ where $0 < \delta < p$. Using the relations in (3),

$$\begin{aligned} x = x_1 &= a_1 \cdots a_q \\ &= (a_1 \cdots a_p)(a_{p+1} \cdots a_{2p}) \cdots (a_{(j-1)p+1} \cdots a_{jp})(a_{jp+1} \cdots a_{jp+\delta}) \\ &= (a_1 \cdots a_p)(ba_1 \cdots a_p b^{-1}) \cdots (b^{j-1} a_1 \cdots a_p b^{1-j})(b^j a_1 \cdots a_\delta b^{-j}) \\ &= (a_1 \cdots a_p b)^j (a_1 \cdots a_\delta) b^{-j}. \end{aligned}$$

Similarly,

$$\begin{aligned} x_2 &= a_2 \cdots a_{q+1} \\ &= (a_2 \cdots a_p)(a_{p+1} \cdots a_{2p}) \cdots (a_{(j-1)p+1} \cdots a_{jp})(a_{jp+1} \cdots a_{jp+\delta+1}) \\ &= (a_2 \cdots a_p b)(a_1 \cdots a_p b)^{j-1}(a_1 \cdots a_{\delta+1}) b^{-j}. \end{aligned}$$

In general, for $1 \leqslant k \leqslant p$,

$$x_k = (a_k \cdots a_p b)(a_1 \cdots a_p b)^{j-1}(a_1 \cdots a_{\delta+k-1}) b^{-j} \qquad \text{if } \delta + k - 1 \leqslant p, \text{ or}$$

$$x_k = (a_k \cdots a_p b)(a_1 \cdots a_p b)^j (a_1 \cdots a_{\delta+k-1-p}) b^{-j-1} \qquad \text{if } \delta + k - 1 > p; \tag{4}$$

$$x_{p+1} = b(a_1 \cdots a_p b)^j (a_1 \cdots a_{\delta-1}) b^{-j-1}.$$

Now we are ready to find a_1, a meridian of $K(p, q)$, in terms of x, y and b. We write $x^\beta = z_1 z_2 \cdots z_\beta$ where each $z_i = a_1 \cdots a_q$ and we make a replacement for each z_i to find a new representation for x^β. Replace z_1 by

$$v_1 = x_1 = (a_1 \cdots a_p b)^j (a_1 \cdots a_\delta) b^{-j}.$$

Replace z_2 by $v_2 = x_{\delta+1}$ where $x_{\delta+1}$ is written as in (4). Note that the last "a" letter in v_1 is a_δ and the first "a" letter in $x_{\delta+1}$ is $a_{\delta+1}$. In general,

(a) if z_i is replaced by

$$v_i = x_k = a_k \cdots a_p b(a_1 \cdots a_p b)^{j-1}(a_1 \cdots a_{\delta+k-1})b^{-j}$$

then z_{i+1} is replaced by $x_{\delta+k}$,

(b) if z_i is replaced by

$$v_i = a_k \cdots a_p b(a_1 \cdots a_p b)^j(a_1 \cdots a_{\delta+k-1-p})b^{-j-1}$$

then z_{i+1} is replaced by $v_{i+1} = x_{\delta+k-p}$, and

(c) if z_i is replaced by

$$v_i = b(a_1 \cdots a_p b)^j(a_1 \cdots a_{\delta-1})b^{-j-1}$$

then z_{i+1} is replaced by $v_{i+1} = x_\delta$, where each of the x_l's is written as in (4). Note that the v_i terms are chosen so that the "a" terms "match up," that is, if the last "a" letter in v_i is a_l then the first "a" letter in v_{i+1} is a_{l+1}. A general formula is $v_i = x_{[(i-1)q]+1}$ where, for m an integer, $[m]$ is defined as follows:

$$[m] = \mu \quad \text{if } m = \sigma p + \mu, 0 < \mu < p,$$

$$[m] = p \quad \text{if } m = \sigma p.$$

Let us examine the product $v_1 v_2 \cdots v_\beta$. If all the b's were deleted it would be

$$(a_1 \cdots a_p)(a_1 \cdots a_p) \cdots (a_1 \cdots a_p)a_1$$

with the group $(a_1 \cdots a_p)$ appearing $(-\alpha)$ times. The last letter is a_1 since the product $x^\beta = (a_1 \cdots a_q)^\beta$ contains βq "a" letters which we have divided into $(-\alpha)$ groups of p plus one. Now let us consider where the various b terms lie in $v_1 \cdots v_\beta$. First, there is a b (with no exponent) after each a_p. Also, there are the b^{-j} and b^{-j-1} terms. Now b and x commute, $v_i = x$ in $\pi_1(T - K(p, q))$, and the terms b^{-j} and b^{-j-1} appear at the end of each v_i. Hence, all terms of this form may be commuted to the right-hand end of the product. Hence,

$$x^\beta = v_1 v_2 \cdots v_\beta = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\mu$$

where b^μ is the product of all the terms of the form b^{-j} or b^{-j-1}.

Next we show $\mu = -\alpha$. Now x^β is homologous in $T - K(p, q)$ to $a_1^{\beta q} = a_1^{1-\alpha p}$. But if we look at $x^\beta = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\mu$ as a word in $H_1(T - K(p, q))$ we get $x^\beta = a_1^{-\alpha p+1} b^{-\alpha+\mu}$; hence $-\alpha + \mu = 0$. Therefore,

$$x^\beta = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\alpha = y^{-\alpha} a_1 b^\alpha.$$

Hence, $a_1 = y^\alpha x^\beta b^{-\alpha}$. Therefore, in case $p < q$, a meridian of $K(p, q)$ is $y^\alpha x^\beta b^{-\alpha}$.

Case 2. $p > q$. As in Case 1, we write each x_k in terms of $a_1, \cdots, a_p$, and b:

$$
\begin{aligned}
&x_k = a_k \cdots a_{k+q-1} \qquad \text{if } k+q-1 \leqslant p, \\
(5) \qquad &x_k = a_k \cdots a_p b a_1 \cdots a_{k+q-1-p} b^{-1} \quad \text{if } p < k+q-1 < p+q; \\
&x_{p+1} = b a_1 \cdots a_q b^{-1}.
\end{aligned}
$$

As in Case 1, we write $x^\beta = z_1 \cdots z_\beta$ where each $z_i = a_1 \cdots a_q$. Again we replace each z_i in this product. z_1 is replaced by $v_1 = x_1 = a_1 \cdots a_q$. In general,

(a) if z_i is replaced by $v_i = x_k = a_k \cdots a_{k+q-1}$ with $k+q-1 \leqslant p$ then z_{i+1} is replaced by $v_{i+1} = x_{k+q}$, written as in (5),

(b) if z_i is replaced by $v_i = a_k \cdots a_p b a_1 \cdots a_{k+q-1-p} b^{-1}$ then z_{i+1} is replaced by $v_{i+1} = x_{k+q-p}$, and

(c) if z_i is replaced by $v_i = x_{p+1} = b a_1 \cdots a_q b^{-1}$ then z_{i+1} is replaced by $v_{i+1} = x_{q+1}$.

Then $x^\beta = v_1 v_2 \cdots v_\beta$ and if all the b's were removed, the product would be $(a_1 \cdots a_p)^{-\alpha} a_1$. Now there is a b after every a_p and there is a b^{-1} at the end of each v_i. As in Case 1, the b^{-1} terms may be commuted to the right-hand end of the product and we have

$$x^\beta = (a_1 \cdots a_p b)^{-\alpha} a_1 b^\alpha = y^{-\alpha} a_1 b^\alpha.$$

Hence, in either case, a meridian of $K(p, q)$ is $c = a_1 = y^\alpha x^\beta b^{-\alpha}$. Now $x^p b^q = y^q$ is a longitude of $K(p, q)$ and $y^q \sim a_1^{pq} b^q$ in $T - K(p, q)$. Let $g = c^{-pq} y^q$. Then $g \sim b^q$, so g is homologous, in $T - K(p, q)$, to a multiple of L.

Now $G(K: r, s)$ is obtained from $\pi_1(T - K(p, q))$ by adding the relation $c^r g^s = 1$, that is, $c^r(c^{-pq} y^q)^s = 1$. Now c is a meridian of K and y^q is a longitude; hence they commute in $\pi_1(T - K(p, q))$. Therefore, the added relation is $c^{r-spq} y^{sq} = 1$ or $(y^\alpha x^\beta b^{-\alpha})^{r-spq} y^{qs} = 1$. In this lemma we are assuming $r - spq = 1$; hence $G(K: r, s)$ has the following presentation:

$$\{b, x, y\colon bx = xb,\ y^q = b^q x^p,\ y^\alpha x^\beta b^{-\alpha} y^{qs} = 1\}. \tag{6}$$

Our next step is to eliminate y from this presentation.

The last relation is $y^\alpha = y^{-qs} b^\alpha x^{-\beta}$ or

$$y^\alpha = (y^q)^{-s} b^\alpha x^{-\beta} = (b^q x^p)^{-s} b^\alpha x^{-\beta} = b^{\alpha - sp} x^{-sp-\beta}.$$

Hence, $y = y^{\alpha p+\beta q} = (y^{\alpha})^p (y^q)^{\beta} = (b^{\alpha - sq}x^{-sp-\beta})^p (b^q x^p)^{\beta} = b^{1-spq}x^{-sp^2}$. Substituting for y in the second relation in (6), we obtain

$$(b^{1-spq}x^{-sp^2})^q = b^q x^p \quad \text{or} \quad b^{-spq^2} = x^{p(1+spq)} = x^{pr}.$$

Substituting for y in $y^{\alpha} = b^{\alpha-sq}x^{-sp-\beta}$, which is the same as the last relation in (4), we obtain $(b^{1-spq}x^{sp^2})^{\alpha} = b^{\alpha-sp}x^{-sp-\beta}$, which reduces to

$$x^{\beta+sp(1-\alpha p)} = b^{sq(\alpha p-1)}, \quad \text{or}$$

$$x^{\beta+\beta spq} = b^{-\beta sq^2}, \quad \text{or}$$

$$x^{\beta r} = b^{-\beta sq^2}.$$

Hence $G(K\colon r, s)$ is isomorphic to

$$\{b, x\colon bx = xb,\ x^{pr} = b^{-spq^2},\ x^{\beta r} = b^{-\beta sq^2}\}. \tag{7}$$

The last two relations imply

$$x^r = (x^{pr})^{\alpha}(x^{\beta r})^q = b^{-\alpha spq^2 - \beta sq^3} = b^{-sq^2(\alpha p+\beta q)} = b^{-sq^2}.$$

But the relation $x^r = b^{-sq^2}$ implies the last two relations in (5). Hence the group in (5) is isomorphic to $\{b, x\colon bx = xb,\ x^r b^{sq^2} = 1\}$.

Now if C is a simple closed curve on Bd T which is homologous (on Bd T) to $rM + sq^2L$ then C is homotopically nontrivial on Bd T but the element of $G(K\colon r, s)$ generated by C is $x^r b^{sq^2}$. Hence C is null-homotopic in $T(K\colon r, s)$ so C is a meridian of $T(K\colon r, s)$.

LEMMA 2(b). *If* $r = 1$ *then*

$$G_d(K(p, q)\colon r, s) = \{b, x\colon bx = xb,\ x^{r+spq}b^{sq^2} = 1\}$$

and a curve homologous to $(r + spq)M + sq^2L$ *is a meridian of the solid torus* $T_d(K\colon r, s)$, *where* x *and* b *are as in Lemma* 2(a). *If* $r = -1$ *then the conclusion is the same with* s *replaced by* $-s$.

This lemma follows from Lemma 2(a) and the remarks which precede Corollary 1.

COROLLARY 5. *If* K *is a* (p, q) *torus knot in* S^3 *and* $|r - spq| = 1$ *then* $M^3(K\colon r, s)$ *is an* $L(sq^2, r)$ *lens space.*

PROOF. From [1, p. 108], a lens space $L(sq^2, r)$ is homeomorphic to a union of two solid tori T_1 and T_2 where $T_1 \cap T_2 = \text{Bd}\, T_1 = \text{Bd}\, T_2$ and such that a meridian of T_1 is sewn to a curve C on Bd T_2 with $C \sim sq^2M_2 + rL_2$ where (M_2, L_2) is a meridian-longitude pair for T_2.

Now the manifold $M^3(K\colon r, s)$ is obtained from S^3 by removing the interior of a regular neighborhood T' of K and sewing in a solid torus T'' such that a meridian of T'' is sewn to a curve on Bd T' homologous to $rc + sg$ where (c, g) is a meridian-longitude pair for T' with $g \sim 0$ in $S^3 - K$. Let T be an unknotted solid torus in S^3 such that $K \subset \text{Int } T$ and K cobounds an annulus with a curve K_1 on Bd T such that $K_1 \sim rc + sg$ on Bd T. Let $T_2 = S^3 - \text{Int } T$. Then the embedding of K in T is of the kind considered in Theorem 2; hence $M^3(K: r, s) = T(K: r, s) \cup T_2$. Since $|r - spq| = 1$, $T(K: r, s)$ is a solid torus. Also, by Lemma 2(a), a meridian of $T(K: r, s)$ is a curve on Bd T homologous to $rM \pm sq^2L$ where (M, L) is a meridian-longitude pair for T such that (L, M) is a meridian-longitude pair for T_2. Therefore, by the remarks of the preceding paragraph, $M^3(K: r, s)$ is the lens space $L(sq^2, r)$.

We conclude this section with two remarks concerning some results of J. Simon [7].

REMARK 1. In Theorem 3 of [7], Simon noted the following: If K is a knot in S^3 and J is a $(1, 2)$ cable about K then $\pi_1(M^3(J: 1, 1)) \cong \pi_1(M^3(K: 1, 4))$.

The results of this section may be used to conclude the following: If T is a regular neighborhood of K in S^3 such that $J \subset \text{Int } T$ and J cobounds an annulus with a $(1, 2)$ curve on Bd T, then

$$M^3(J: 1, 1) = T(J(1, 1)) \cup (S^3 - \text{Int } T).$$

Now in this case, $r - spq = -1$; hence $T(J: 1, 1)$ is a solid torus by Corollary 3. By Lemma 2, a meridian of $T(J: 1, 1)$ is a curve on Bd T homologous to $M + 4L$ where (M, L) is a meridian-longitude pair for T such that $L \sim 0$ in $S^3 - \text{Int } T$. Therefore, we may conclude that the two fundamental groups mentioned above are isomorphic because the two surgery manifolds $M^3(J: 1, 1)$ and $M^3(K: 1, 4)$ are homeomorphic.

REMARK 2. In Theorem 4 of [7], Simon noted that a simply connected 3-manifold can be obtained by nontrivial surgery on a link $K \cup J$ in S^3 where K is a nontrivial knot and J is a cable about K. The results of this section can be used to conclude two things in this regard: (1) Any homotopy 3-sphere obtained by this kind of surgery can also be obtained by surgery on K, and (2) S^3 can be obtained by nontrivial surgery on such a link.

First, we describe the construction in detail. Let K be a knot in S^3 and suppose J is a (p, q) cable about K. This means there is a solid torus T^* in S^3 with a meridian-longitude pair (M^*, L^*) such that:

(i) there is a simple closed curve J^* in Int T^* which cobounds an annulus with a curve J_1^* on Bd T^* such that $J_1^* \sim pM^* + qL^*$, and

(ii) there is a homeomorphism f of T^* into S^3 such that f takes a core of T^* onto K, $f(J^*) = J$ and $f(L^*) \sim 0$ in $S^3 - \operatorname{Int} f(T^*)$.

Let $T = f(T^*)$, $M = f(M^*)$, $L = f(L^*)$ and $J_1 = f(J_1^*)$. Let T_K be a regular neighborhood of K in $\operatorname{Int} T$ such that $T_K \cap J = \varnothing$. Let (c_K, g_K) be a meridian-longitude pair for T_K such that $g_K \sim f(L)$ in $T - K$. Let T_J be a regular neighborhood of J in $\operatorname{Int} T$ such that $T_J \cap T_K = \varnothing$. Let (c_J, g_J) be a meridian-longitude pair for T_J such that $g_J \sim J_1$ in $T - (K \cup J)$.

Now suppose (r_1, s_1) and (r_2, s_2) are pairs of relatively prime integers. Let $M^3 = M^3(K: r_1, s_1 | J: r_2, s_2)$ denote the manifold obtained by (r_1, s_1)-surgery on K and (r_2, s_2)-surgery on J. That is M^3 is obtained from S^3 by removing T_K and T_J and sewing in solid tori T_K'' and T_J'' where a meridian of T_K'' is sewn to a curve homologous to $r_1 c_K + s_1 g_K$ and a meridian of T_J is sewn to a curve homologous to $r_2 c_J + s_2 g_J$. We consider the surgery on K first. Let $T_1 = (T - \operatorname{Int} T_K) \cup T_K''$. T_1 is a solid torus since it is a union of a solid torus, namely T_K'', and a boundary collar. Also a curve M_1 on $\operatorname{Bd} T$ with $M_1 \sim r_1 M + s_1 L$ is a meridian of T_1 and J is a nicely embedded torus knot in T_1.

Now r_1 and s_1 are relatively prime; hence there are integers α and β such that $\alpha r_1 + \beta s_1 = 1$. Let L_1 be a simple closed curve on $\operatorname{Bd} T_1 = \operatorname{Bd} T$ such that $L_1 \sim \beta M - \alpha L$. We claim that L_1 is a longitude of T_1. First, $M \sim \alpha M_1 + s_1 L_1$ and $L \sim \beta M_1 - r_1 L_1$ so M_1 and L_1 generate the homology group $H_1(\operatorname{Bd} T_1)$. It remains to show that the algebraic intersection number $\#(M_1, L_1) = \pm 1$. But $\#(M, M_1) = s_1$ and $\#(L, M_1) = -r_1$; hence $\#(L_1, M_1) = \#(\beta M - \alpha L, M_1) = \beta s_1 - \alpha(-r_1) = 1$. Hence (M_1, L_1) is a meridian-longitude pair for the solid torus T_1.

Now, $J_1 \sim pM + qL \sim (\alpha p + \beta q)M_1 + (ps_1 - qr_1)L_1$; hence J is an $(\alpha p + \beta q, ps_1 - qr_1)$ torus knot in T. Let $p' = \alpha p + \beta q$ and $q' = ps_1 - qr_1$. Let T_2 denote the result of (r_2, s_2) on the knot J in T_1. That is, $T_2 = (T_1 - \operatorname{Int} T_J) \cup T_J''$.

Case 1. If $r_2 \neq \pm 1$, then M^3 is not a homotopy 3-sphere. If M^3 were simply connected then, by Dehn's lemma, once closed complementary domain of the torus $\operatorname{Bd} T$ would be a homotopy solid torus. But one complementary domain is $S^3 - \operatorname{Int} T$ and the other is T_2, neither of which is a homotopy solid torus.

Case 2. $r_2 = \pm 1$. Then T_2 is a solid torus and, by Lemma 2(b), a meridian of T_2 is a curve M_2 on $\operatorname{Bd} T = \operatorname{Bd} T_2$ such that $M_2 \sim (r_2 + s_2 p'q')M_1 + s_2(q')^2 L_1$. Letting $p' = \alpha p + \beta q$, $q' = ps_1 - qr_1$, $M_1 = r_1 M + s_1 L$ and $L_1 = \beta M - \alpha L$ we obtain, using the equation $\alpha r_1 + \beta s_1 = 1$,

$$M_2 \sim (r_1 r_2 + p^2 s_1 s_2 - pqr_1 s_2)M + (r_2 s_1 + pqs_1 s_2 - q^2 r_1 s_2)L.$$

Therefore, if $r_2 = \pm 1$, the surgery manifold M^3 is homeomorphic to $M^3(K: r_1r_2 + p^2s_1s_2 - pqrs_1s_2, r_2s_1 + pqs_1s_2 - q^2r_1s_2)$.

Hence conclusion (1) follows from Case 1 and Case 2. Also it follows from Case 2 that $M^3 \approx S^3$ if

$$r_2 = \pm 1,$$

$$r_1r_2 + p^2s_1s_2 - pqr_1s_2 = \pm 1, \quad \text{and}$$

$$r_2s_1 + pqs_1s_2 - q^2r_1s_2 = 0.$$

4. **Doubly twisted knots.** Before defining doubly twisted knots and considering surgery on them, we describe a method of presenting the group of knots "with twists." Suppose C is the cube $[0, 1] \times [0, 1] \times [0, 1]$, A and B are parallel rectilinear spanning segments of C and T is an unknotted solid torus in $C - (A \cup B)$. See Figure 2a. Suppose T has a longitude g which bounds a disk D in $C - \operatorname{Int} T$ such that D intersects each of A and B in a single interior point. Let c be a meridian of T. Suppose $\operatorname{Int} T$ is removed from C and a solid torus T_1 is sewn in with a meridian of T_1 sewn to a simple closed curve homologous to $c + ng$. Then there is a homeomorphism of $(C - \operatorname{Int} T) \cup T_1$ onto C which is the identity on $\operatorname{Bd} C$ and which takes A and B onto arcs A' and B', as in Figure 2b, where A' and B' have $2n$ crossings. We construct this homeomorphism as follows:

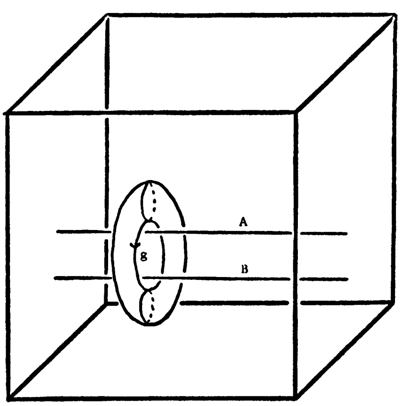

FIGURE 2a

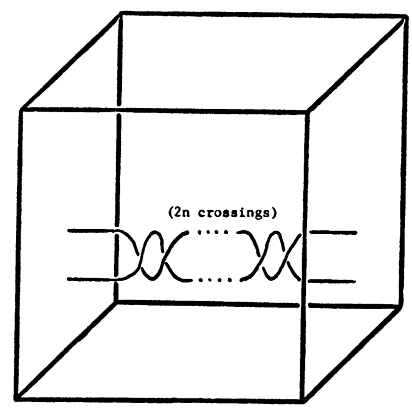

FIGURE 2b

Let h be a homeomorphism of $D \times [-1, 1]$ into $C - \operatorname{Int} T$ such that $h(D \times \{0\}) = D$ and $h(\operatorname{Bd} D \times [-1, 1]) \subset \operatorname{Bd} T = \operatorname{Bd} T_1$. Let f be a homeomorphism of $h(D \times [-1, 1])$ onto itself which is the identity on $D \times \{-1, 1\}$ and which gives one end of $h(D \times [-1, 1])$ n full twists in the $+g$ direction. Let f be the identity on $C - (h(D \times [-1, 1]) \cup \operatorname{Int} T)$. Since f

takes a meridian of T onto a curve homologous to $c + ng$, f may be extended to take C onto $(C - \mathrm{Int}\, T) \cup T_1$. Then f^{-1} is the required homeomorphism.

The action of f^{-1} on the pair (A, B) may be visualized as follows: f^{-1} cuts the pair in the middle, gives them n full twists in the $-g$ direction and pastes them back together.

The twisting homeomorphism f^{-1} provides a connection between certain knot groups and groups obtained by adding a surgery relation to an appropriate link group. For example, the group of the $2n$-twist knot, Figure 3a, may be presented as the group of the link $K \cup J$ in Figure 3b with the added relation $a(y^{-1}x)^{-n} = 1$.

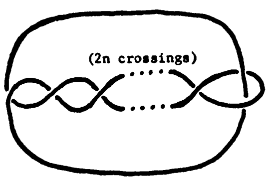

FIGURE 3a

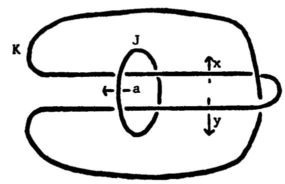

FIGURE 3b

In the remainder of this section, $T(m, n)$ will denote the doubly twisted knot in the solid torus T as shown in Figure 4. Here $n \neq 0$, but m is any integer. Also, if n is odd then m must be even, since otherwise we would have two curves.

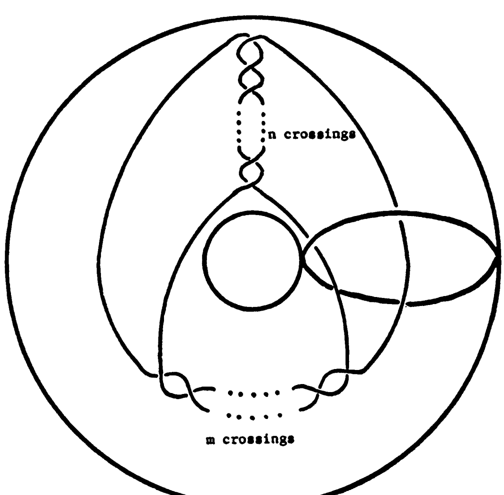

FIGURE 4

THEOREM 3. *For* $(m, n) \neq$ *(even,* ± 1*),* $T(T(m, n)\colon r, s)$ *is a homotopy solid torus only in the trivial case* $(r, s) = (\pm 1, 0)$.

Note that in case $(m, n) = (\text{even}, \pm 1)$ the knot $T(m, n)$ is a torus knot as considered in §3.

PROOF OF THEOREM 3. We consider three cases: (1) $(m, n) = (\text{even}, \text{even})$, (2) $(m, n) = (\text{odd}, \text{even})$, and (3) $(m, n) = (\text{even}, \text{odd})$ with $n \neq \pm 1$. In each case we show that the fundamental group of $T(T(m, n): r, s)$ is not infinite cyclic.

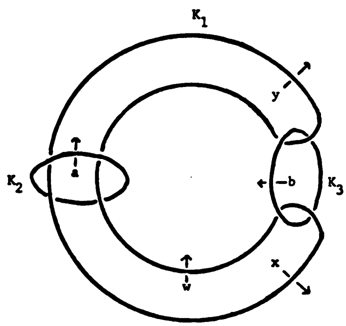

FIGURE 5

Case 1. Both m and n are even. For a fixed even integer n, the manifolds $T(m, n)$, m even, are equivalently embedded in T. Hence we consider only the case $m = 0$. To calculate $\pi_1(T - T(m, n))$ we consider the link in Figure 5. The group $\pi_1(T - T(m, n))$ will be calculated as $\pi_1(T - (K_1 \cup K_2 \cup K_3))$ with the added relation $b(yx^{-1})^{-k} = 1$, where $n = 2k$. The added relation comes from surgery on K_3 and gives the knot the $2k$ crossings. Hence $\pi_1(T - T(0, n))$ has a presentation with generators a, b, w, x and y and relations:

$$(1)\ a^{-1}wa = b^{-1}yb, \quad (2)\ xax^{-1} = waw^{-1}, \quad (3)\ y^{-1}by = x^{-1}bx,$$

$$(4)\ y = a^{-1}xa, \qquad (5)\ w = b^{-1}xb, \quad \text{and} \quad (6)\ b = (yx^{-1})^k.$$

Now (3) is a consequence of (6). We add a generator z and the relation $z = yx^{-1}$ and then we eliminate y. Then, using (6), we eliminate b. We obtain a presentation with generators a, w, x and z and relations:

$$(1')\ a^{-1}wa = z^{1-k}xz^k, \qquad (2')\ xax^{-1} = waw^{-1},$$

$$(4')\ zx = a^{-1}xa, \qquad (5')\ w = z^{-k}xz^k.$$

Now (5′), (4′) and (2′) imply (1′); hence $\pi_1(T - T(0, n))$ has the presentation

$$(7) \qquad \{a, w, x, z: z = a^{-1}xax^{-1} = a^{-1}waw^{-1}, w = z^{-k}xz^k\}.$$

Now a meridian of $K_3 = T(0, n)$ is $c = w$ and a longitude is $g = ab^{-1}a^{-1}b = az^{-k}a^{-1}z^k$. Note that $g \sim 0$ in $T - T(0, n)$. Then the surgery manifold

$T(T(0, n)\colon r, s)$ has fundamental group $G(T(0, n)\colon r, s)$ which is obtained from (7) by adding the relation $c^r g^s = 1$ or $w^r(az^{-k}a^{-1}z^k)^s = 1$. If this presentation of $G(T(m, n)\colon r, s)$ is abelianized, we obtain the group $\{a, w\colon aw = wa, w^r = 1\}$ which is not infinite cyclic unless $r = \pm 1$. Hence we assume $r = -1$. (The case $r = 1$ is essentially the same.) Hence we have the group

(8) $\{a, w, x, z\colon z = a^{-1}xax^{-1} = a^{-1}waw^{-1}, w = z^{-k}xz^k, w = (az^{-k}a^{-1}z^k)^s\}$.

We eliminate w using the third relation and obtain relations

(9) $$z = a^{-1}xax^{-1},$$

(10) $$z = a^{-1}z^{-k}xz^k az^{-k}x^{-1}z^k, \quad \text{and}$$

(11) $$z^{-k}xz^k = (az^{-k}a^{-1}z^k)^s \quad \text{or} \quad x = (z^k az^{-k}a^{-1})^s.$$

Using (11), we eliminate x and obtain:

(9′) $$z = a^{-1}(z^k az^{-k}a^{-1})^s a(z^k az^{-k}a^{-1})^{-s} = (a^{-1}z^k az^{-k})^s(z^k az^{-k}a^{-1})^{-s},$$

and

$$z = (z^{-k}a^{-1}z^k a)^s(az^{-k}a^{-1}z^k)^{-s} \quad \text{or}$$

(10′) $$z = z^k(z^{-k}a^{-1}z^k a)^s z^{-k}z^k(az^{-k}a^{-1}z^k)^{-s}z^{-k} \quad \text{or}$$

$$z = (a^{-1}z^k az^{-k})^s(z^k az^{-k}a^{-1})^{-s}.$$

These calculations show (9′) and (10′) are the same; hence $G(T(0, n)\colon -1, s)$ has the presentation

$$\{a, z\colon z = (a^{-1}z^k az^{-k})^s(z^k az^{-k}a^{-1})^{-s}\}.$$

We obtain a quotient group by adding the relation $a^2 = 1$. The quotient group is $\{a, z\colon a^2 = 1, z = (az^k az^{-k})^{2s}\}$. The second relation is $z^{2ks+1} = (az^k a)^{2s} = az^{2ks}a$ or $az^{2ks+1} = z^{2ks}a$. Hence the quotient group is

(12) $$\{a, z\colon a^2 = 1, az^{2ks+1} = z^{2ks}a\}.$$

Now the relations in (12) imply $z^{2ks+1}az = az^{2ks+1}$, or $z^{2ks+1}az = z^{2ks}a$, or $zaz = a$, or $(az)^2 = 1$. The relations in (12) and this last relation imply $aza = z^{-1}$ which implies $az^{2ks}a = z^{-2ks}$. Using the relations in (12), we change this to $az^{2ks}a = az^{-2ks-1}a$ or $z^{4ks+1} = 1$. Hence the relations in (12) imply $a^2 = (az)^2 = z^{4ks+1} = 1$. Now these last relations imply $aza = z^{-1}$ which implies $az^{2ks}a = z^{-2ks}$, or $az^{2ks}a = z^{2ks+1}$, or $az^{2ks} = z^{2ks+1}a$. Hence, the group (12) is the same as $\{a, z\colon a^2 = (az)^2 = z^{4ks+1} = 1\}$. But this is a presentation of the dihedral group $\mathfrak{D}_{4ks+1}$. See [5, p. 6]. Since $|4ks + 1| \neq 1$, $G(T(0, n)\colon -1, s)$ is nonabelian because it has the nonabelian quotient $\mathfrak{D}_{4ks+1}$.

Case 2. m is odd and n is even. Say $n = 2k$. As in Case 1, for fixed n, the knots $T(m, n)$, m odd, are equivalently embedded in T. Hence we consider only the case $m = 1$. We find a presentation for $\pi_1(T - T(1, n))$ by considering the link in Figure 6 and adding a relation corresponding to surgery on K_3. Hence we have a presentation with generators a, b, w, x and y and relations:

(13) $a^{-1}wa = b^{-1}yb$, (14) $xax^{-1} = waw^{-1}$, (15) $y^{-1}by = wbw^{-1}$,

(16) $x = aya^{-1}$, (17) $w^{-1}xw = b^{-1}wb$, (18) $b(yw)^{-k} = 1$.

Now (15) is a consequence of (18). We add a generator z and the relation $z = yw$ and then eliminate y. Next we eliminate b using (18) and then x using (16). We are left with generators a, w, and z and the following relations:

$$(13') \qquad a^{-1}wa = z^{1-k}w^{-1}z^k,$$

$$(14') \qquad azw^{-1}awz^{-1}a^{-1} = waw^{-1}, \quad \text{and}$$

$$(17') \qquad w^{-1}azw^{-1}a^{-1}w = z^{-k}wz^k.$$

Now (17′) is a consequence of (13′) and (14′); hence $\pi_1(T - T(1, n))$ has the presentation

$$(19) \qquad \{a, w, z: a^{-1}wa = z^{1-k}w^{-1}z^k, waw^{-1} = azw^{-1}awz^{-1}a^{-1}\}.$$

Now a meridian of $T(1, n)$ is $c = w$ and a natural longitude, from Figure 6, is $g_1 = ab^{-1}a^{-1}wb^{-1} = az^{-k}a^{-1}wz^{-k}$. Hence a longitude which is null-homologous in $T - T(1, n)$ is $g = w^{4k-1}az^{-k}a^{-1}wz^{-k}$. Therefore, $G(T(1, n): r, s)$

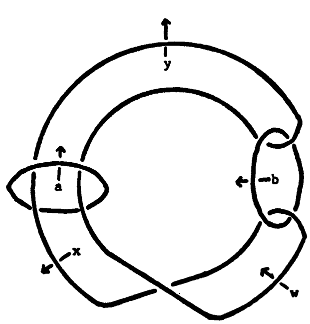

FIGURE 6

is found by adding to (19) the relation $c^rg^s = 1$ or $w^{r+4ks-s}(az^{-k}a^{-1}wz^{-k})^s = 1$. Now if $G(T(1, n): r, s)$ is abelianized, we obtain the group $\{a, w: aw = wa, w^r = 1\}$; hence we may assume $|r| = 1$. Since the two cases are essentially the same, we assume $r = 1$. Then $G(T(1, n): 1, s)$ has the presentation

(20) $\{a, w, z: a^{-1}wa = z^{1-k}w^{-1}z^{k},\ waw^{-1} = azw^{-1}awz^{-1}a^{-1},$

$$w^{4ks-s+1} = (az^{-k}a^{-1}wz^{-k})^{s}\}.$$

We obtain a quotient group by adding the relations $z^k = 1$ and $a^2 = 1$. Then we use the first relation in (20) to eliminate z. This yields the group

(21) $$\{a, w: a^2 = (aw)^{2k} = w^{4ks+1} = 1\}.$$

This group has the dihedral group $\mathfrak{D}_{4ks+1}$ as a quotient; hence, as in Case 1, $G(T(1, n): r, s)$ is nonabelian.

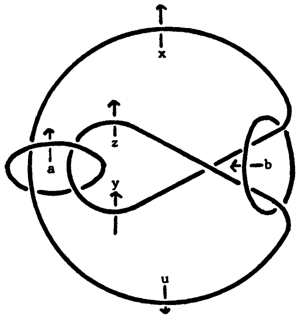

FIGURE 7

Case 3. m is even, n is odd and $n \neq \pm 1$. As in the first two cases, we consider only the case $m = 0$ and we find a presentation for $\pi_1(T - T(0, n))$ by considering the link in Figure 7 and adding a surgery relation. Since a reflection of T takes $T(0, -n)$ onto $T(0, n)$, we assume $n > 0$. Hence we have a presentation with generators a, b, u, x, y and z and relations

(22) $uau^{-1} = y^{-1}ay$, (23) $x = a^{-1}ua$, (24) $y = aza^{-1}$,

(25) $b^{-1}xb = z^{-1}yz$, (26) $z = b^{-1}ub$, (27) $x^{-1}bx = u^{-1}bu$, and

(28) $b(xu^{-1})^{-k} = 1$

where $n = 2k + 1$ (hence $k > 0$). Now (27) is a consequence of (28). We add a new generator t with $t = xu^{-1}$ and then eliminate x. Then, using (28), we eliminate b, and using (24), we eliminate y. We are left with the following relations:

(22′) $$uau^{-1} = az^{-1}aza^{-1},$$

(23′) $$t = a^{-1}uau^{-1},$$

(25′) $$t^{1-k}ut^{k} = z^{-1}aza^{-1}z, \quad \text{and}$$

(26′) $$z = t^{-k}ut^{k}.$$

Now (25′) is a consequence of (22′), (23′) and (26′). Also, we may eliminate

z using (26). We are left with generators a, u, and t and relations

(22″) $$uau^{-1}at^{-k}u^{-1}t^{k}at^{-k}ut^{k}a^{-1}, \text{ and}$$

(23″) $$t = a^{-1}uau^{-1}.$$

Now, from Figure 7, a meridian of $T(0, n)$ is $c = u$ and a longitude is $g = uabz^{-1}ab^{-1} = uau^{-1}t^{k}at^{-k}$. Note that $g \sim a^{2}$, and a is a longitude of the solid torus T. Hence, $G(T(0, n): r, s)$ has the presentation

(29) $$\{a, t, u: uau^{-1} = at^{-k}u^{-1}t^{k}at^{-k}ut^{k}a^{-1}, t = a^{-1}uau^{-1}, u^{r+s}(au^{-1}t^{k}at^{-k})^{s} = 1\}.$$

We now obtain some quotient groups. Adding the relation $t^{k} = 1$ and eliminating t we have

(30) $$\{au: uau^{-1} = au^{-1}aua^{-1}, (a^{-1}uau^{-1})^{k} = 1, u^{r+s}(au^{-1}a)^{s} = 1\}.$$

Now the first relation is $a^{-1}ua^{-1}u = ua^{-1}ua^{-1}$ which is the same as $(ua^{-1})^{2}u = u(ua^{-1})^{2}$. Using this, the third relation is $u^{r+s} = (a^{-1}ua^{-1})^{s}$ or $u^{r+s} = [u^{-1}(ua^{-1})^{2}]^{s}$ or $u^{r+2s} = (ua^{-1})^{2s}$. Therefore, adding the relation $(ua^{-1})^{2} = 1$ to (30), we obtain the quotient group

$$G_1 = \{a, u: (ua^{-1})^{2} = (a^{-1}uau^{-1})^{k} = u^{r+2s} = 1\}.$$

In terms of u and d with $d = au^{-1}$,

$$G_1 = \{d, u: d^{2} = u^{r+2s} = (u^{-1}d^{-1}ud)^{k} = 1\}.$$

Next we obtain another quotient group of $G(T(0, n): r, s)$ by adding the relation $t^{k+1} = 1$ to (29). After eliminating t, we have the group

(31) $$\{a, u: (a^{-1}uau^{-1})^{k+1} = 1, a^{-1}u^{-1}auaua^{-1}u^{-1} = 1, u^{r+s}[u^{-1}auau^{-1}]^{s} = 1\}.$$

The second relation is $uaua = auau$ which is the same as $(ua)^{2}u = u(ua)^{2}$. Using this last relation, we may change the third relation to $u^{r-2s}(ua)^{2s} = 1$. Finally we add the relation $(ua)^{2} = 1$ to (31) and obtain the group

$$G_2 = \{a, u: u^{r-2s} = (ua)^{2} = (a^{-1}uau^{-1})^{k+1} = 1\}.$$

In terms of u and d with $d = ua$,

$$G_2 = \{d, u: d^{2} = u^{r-2s} = (d^{-1}udu^{-1})^{k+1} = 1\}.$$

Now the groups G_1 and G_2 are both instances of the group

(32) $$\{S, T: S^{l} = T^{2} = (S^{-1}T^{-1}ST)^{p} = 1\}.$$

From [3], this group is dihedral if $|l| = 2$ and $|p| > 1$, is the direct product of A_4 and a cyclic group if $|l| = 3$ and $|p| = 2$ and is infinite in all other cases with $|l| > 1$ and $|p| > 1$. Hence (32) is nonabelian unless $|l| = 1$ or $|p| = 1$. Therefore, since k is positive, G_1 is nonabelian unless $k = 1$ or $|r + 2s| = 1$ and G_2 is nonabelian unless $|r - 2s| = 1$. Hence, the groups G_1 and G_2 may be used to show $G(T(0, n)\colon r, s)$ is nonabelian except for the case $k = 1$ and $|r - 2s| = 1$.

Now for the case $k = 1$, from (29), $G(T(0, n)\colon r, s)$ is presented by

$$(33) \quad \{a, t, u\colon t^2 = u^{-1}tat^{-1}uta^{-1}, t = a^{-1}uau^{-1}, u^{r+s}(au^{-1}tat^{-1})^s = 1\}.$$

Now the first relation is $u^{-1}tat^{-1} = t^2at^{-1}u^{-1}$. Substituting this into the third relation we obtain $u^r(at^2at^{-1})^s = 1$. (Recall that u and at^2at^{-1} commute since u is a meridian and at^2at^{-1} is derived from a longitude.) Now, by adding the relations $t^3 = (at^{-1})^2 = 1$, we obtain the quotient group

$$(34) \quad \{a, t, u\colon t^{-1} = u^{-1}tat^{-1}uta^{-1}, t = a^{-1}uau^{-1}, u^r = t^3 = (at^{-1})^2 = 1\}.$$

Now the first relation is $t^{-1}at^{-1}u^{-1} = u^{-1}tat^{-1}$. Substituting ta^{-1} for at^{-1} twice ($ta^{-1} = at^{-1}$ is a consequence of $(at^{-1})^2 = 1$), the first relation in (33) becomes $t^{-1} = ua^{-1}u^{-1}a$, which is the same as the second relation in (33). Therefore, the first relation in (33) is a consequence of the others. Then, after eliminating t, the group (33) becomes

$$(35) \qquad \{a, u\colon u^r = (a^{-1}uau^{-1})^3 = (aua^{-1}u^{-1}a)^2 = 1\}.$$

If we add to this the relation $a^2 = 1$, which implies the third relation in (35), we have the group

$$G_3 = \{a, u\colon u^r = (a^{-1}uau^{-1})^3 = a^2 = 1\}.$$

Now G_3 is nonabelian unless $r = \pm 1$. Hence, the groups G_2 and G_3 may be used to show $G(T(0, 1)\colon r, s)$ is nonabelian unless $|r| = 1$ and $|r - 2s| = 1$. Solving simultaneously, we find we are left with just one case, $r = s = 1$. Now from (33) and the remarks just after (33), $G(T(0, 1)\colon 1, 1)$ has the presentation

$$(36) \quad \{a, t, u\colon t^2 = u^{-1}tat^{-1}uta^{-1}, t = a^{-1}uau^{-1}, u = ta^{-1}t^{-2}a^{-1}\}.$$

Eliminating u, we find that the remaining two relations are the same; hence (35) is equivalent to

$$\{a, t\colon t^2a^{-1}t^{-2}a^{-1}t^2 = a^{-1}ta^{-1}\}.$$

In terms of t and d, with $d = a^{-1}t$, this is

$$\{d, t\colon t^2dt^{-3}dt^2 = d^2\}.$$

Adding the relations $t^3 = d^4 = (dt)^4 = 1$ we obtain the quotient

$$\{d, t: d^4 = t^3 = (dt)^4 = (d^2t)^2 = 1\}.$$

Now from §1.3 of [4], this is a presentation of the polyhedral group $(4, 4|3, 2)$ which is known to be nonabelian. This concludes the proof of Theorem 3.

COROLLARY. *If K is a knot in S^3 which has a doubly twisted knot $T(m, n)$ as a companion, with $(m, n) \neq$ (even, ± 1), then K has property* P.

PROOF. Let $T(m, n)$ be the doubly twisted knot in the solid torus T as shown in Figure 4. K has $T(m, n)$ as a companion means there is an embedding h of T into S^3 such that $h(T)$ is knotted and $h(T(m, n)) = K$. If K does not have property P then, for some pair (r, s) of relatively prime integers, $M^3(K: r, s)$ is simply connected. Hence, by Dehn's lemma, one closed complementary domain of $h(\mathrm{Bd}\, T)$ is a homotopy solid torus. But one closed complementary domain is $\mathrm{Cl}(S^3 - h(T))$ which is a cube with a knotted hole and the other is $T(T(m, n): r, s)$, neither of which is a homotopy solid torus. Hence K has property P.

The class of knots considered in the corollary includes all doubled knots, which were shown, except for zero twists, to have property P by Bing and Martin [2].

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